Complex analysis on *o*-minimal structures

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1 Topological fields

1.1 Topological fields

Definition 1.1. *A topological field is a field F together with a topology τ on F for which the sum, product, additive and multiplicative inverse functions are continuous.*

Properties A few properties:

- The space (F,τ) is Hausdorff.
- If (F,τ) is compact, then F is finite.
- $x\mapsto x^{-1}$ has no limit at 0.
- There exists a non-empty open proper subset of F.

Example 1.2. The following are topological fields.

- Finite fields with the discrete topology.
- Linearly ordered fields with the order topology.
- If F is any field, then the field $F[[t^{\mathbb{Z}}]]=F((t))$ of formal Laurent series with the valuation topology (in fact any valued field). Note that $F\subseteq F((t))$ is closed, and that the induced topology on F is the discrete topology.
- Let (F,τ) be a topological field, and consider an algebraic extension $F[\alpha]$. Then $F[\alpha]\simeq F^n$ as a vector space over F. The product topology then induces a topological field.

1.2 Differentiability in topological fields

We fix a topological field (F,τ).

Definition 1.3. *Let U be open, let $x_0\in U$ and let $f:U\longrightarrow F$. Then f is said differentiable if there exists a $d\in F$ such that*

$$\lim_{\substack{h\to 0\\ h\neq 0}}\frac{f(x_0+h)-f(t)}{h}=d.$$

The number d is unique, and we write $d:=f'(x_0)$.

This is preserved by sums, products, quotients, composition and so on. Moreover differentiability implies continuity. We also have the Cauchy-Riemann equations.

We will next consider more specific contexts: *o*-minimality in particular.

1.3 The *o*-minimal case

Let $\mathcal{R}=(R,+,\times,<,\text{—})$ be *o*-minimal and write $\mathcal{K}=(K,+,\times)$ for its algebraic closure $K=R[\sqrt{-1}]$. Any K-rational function is differentiable.

Consider the case of $\mathbb{R}_{\mathrm{an}}$. If $h\in\mathbb{C}[[z]]$ converges in a neighborhood of zome $z_0\in\mathbb{C}$, then there is some open $U\ni z_0$ such that $h\restriction U$ is definable in $\mathbb{R}_{\mathrm{an}}$.

On $\mathbb{C}$, the function exp is not definable in any o-minimal expansion of $\mathbb{R}$. Fact: let U be open such that $\exp\upharpoonright U$ is definable in some o-minimal expansion of $\mathbb{R}$. Then the imaginary part of U must be bounded. Indeed, defining $g(z)=\frac{\exp(z)}{|\exp(z)|}=\mathrm{e}^{i\mathrm{Im}(z)}$ on U, and the set $\{(0,y)\in U\cap\{0\}\times i\,\mathbb{R}: \mathrm{e}^{iy}=1\}\subseteq 2\,\pi\,\mathbb{Z}$ (project onto the y-axis if you must) must be bounded by o-minimality. Conversely, in $\mathbb{R}_{\mathrm{an,exp}}$, each such $\exp\upharpoonright U$ is definable whenever U is definable. In particular exp is defined and injective on the strip of $z\in\mathbb{C}$ with $\mathrm{Im}(z)\in[-\pi,\pi)$. So $\log\colon\mathbb{C}\setminus\mathbb{R}^{\geqslant 0}$ is definable.

Question 1. Assume $V\subseteq\mathbb{R}^n$ is open and $f\colon V\longrightarrow\mathbb{R}^n$ is real-analytic and definable in some o-minimal expansion of $\mathbb{R}$, then when can f be extended definably to some $\hat{f}\colon U\longrightarrow\mathbb{C}^n$ which is complex-analytic, where $U\subseteq\mathbb{C}^n$ is open? This is open for f definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Answer 1. Tobias Kaiser proved the result for unary functions definable in $\mathbb{R}_{\mathrm{an,exp}}$. Andre Opris shows in his PhD thesis that this is the case for a large class of functions called restricted exp-log-analytic functions, which is a proper subset of the set of definable functions in $\mathbb{R}_{\mathrm{an,exp}}$.

1.4 Diverging series

Consider the field $\mathcal{R}_{\mathrm{Pui}}=\bigcup_{n>0}\mathbb{R}\big(\big(t^{\frac{1}{n}}\big)\big)$ of formal Puiseux series over $\mathbb{R}$, where t is a positive infinitesimal. The field $\mathcal{R}_{\mathrm{Pui}}$ is real-closed. Let $\mathcal{K}_{\mathrm{Pui}}$ denote its algebraic closure. For all $h\in\mathbb{R}[[z_1,\ldots,z_n]]$ (formal power series), we have a function $h\colon(-t,t)^n\longrightarrow\mathcal{R}_{\mathrm{Pui}}$. Write $\mathcal{R}$ for the structure $(\mathcal{R}_{\mathrm{Pui}},+,\times,<,(h)_{h\in\mathbb{R}[[z_1,\ldots,z_n]]})$. Robinson and Lipshitz showed that this is o-minimal. Any such function $h\in\mathbb{R}[[z_1]]$ can be dedinably extended to $(-t,t)^2\subseteq\mathcal{K}_{\mathrm{Pui}}$.

2 Topological analysis

We fix an o-minimal expansion $\mathcal{R}$ of a real-closed field R, and write K for the algebraic closure of R.

2.1 Winding numbers

Everything will be definable.

Definition 2.1. *A* **definable closed curve** *in R^2 is a definable and continuous $\sigma\colon[0,1]\longrightarrow R^2$ with $\sigma(0)=\sigma(1)$. We sometimes write $\mathcal{C}:=\sigma([0,1])$. The curve σ is called* **simple** *if $\sigma_{[0;1)}$ is injective.*

Example 2.2. $\mathbb{S}^1:=\{z\in K:|z|^2=1\}$. We fix some simple counter-clockwise semi-algebraic parametrization s_0 of $\mathbb{S}^1$.

Given any closed curve $(\sigma,\mathcal{C})$ and a definable continuous map $f\colon\mathcal{C}\longrightarrow\mathbb{S}^1$. We have maps

$$[0,1)\xrightarrow{\sigma}\mathcal{C}\xrightarrow{f}\mathbb{S}^1\xrightarrow{s_0^{\mathrm{inv}}}[0,1)$$

choosing s_0 so that $s_0(0):=f(\sigma(0))$. Write $\hat{f}:=s_0^{\mathrm{inv}}\circ f\circ\sigma\colon[0,1)\longrightarrow[0,1)$. For simplicity, assume that f is not locally constant anywhere. The only possible discontinuities lie in $\hat{f}^{-1}(\{0\})$, which by o-minimality, is finite. Write $0=a_0<\cdots<a_{n-1}<1$ be the discontinuities. We add $a_n:=1$ as a point, in order to define the winding number of f as follows. For $i\in\{0,\ldots,n\}$, write

$$\begin{aligned} f(a_i^+) &:= \lim_{\substack{t\to a_i\\ t>0}} f(t),\\ f(a_i^-) &:= \lim_{\substack{t\to a_i\\ t<0}} f(t). \end{aligned}$$

We define the *winding number* $W_{\mathcal{C}}(f)$ of f as

$$W_{\mathcal{C}}(f):=\sum_{i=0}^{n-1} f(a_{i+1}^-)-f(a_i^+)\in\mathbb{Z}.$$

xample

Example 2.3. Take $f: \mathbb{S}^1 \longrightarrow \mathbb{S}^1; z \mapsto z^2$. Then $W_{\mathbb{S}^1}(f) = 2$. Also $W_{\mathcal{C}}(z \mapsto -z^2) = 2$, and $W_{\mathcal{C}}(z \mapsto z^{-1}) = -1$.

Remark 2.4. If $(f(t))_{t \in D}$ is a definable family of continuous functions $\mathcal{C}_t \longrightarrow \mathbb{S}^1$. Then by o-minimality the numbers n_t corresponding to n above are bounded, so the definition of $W_{\mathcal{C}}(f)$ can be done uniformly.

Let us give a few properties of those winding numbers.

i. If $f: \mathcal{C} \longrightarrow \mathbb{S}^1$ is not surjective, then $W_{\mathcal{C}}(f) = 0$.

ii. Let $f, g: \mathcal{C} \longrightarrow \mathbb{S}^1$ be definably holomorphic. Assume that f and g are homotopy-equivalent, i.e. there is a continuous $H: [0,1]^2 \longrightarrow \mathbb{S}^1$ with $H(0,-) = f$ and $H(1,-) = g$. Then $W_{\mathcal{C}}(f) = W_{\mathcal{C}}(g)$. This relies on the facts that $[0,1]$ is dedinably connected in R, and that the function $t \mapsto W_{\mathcal{C}}(H(t,-))$ is locally constant.

iii. For continuous definable $f, g: \mathcal{C} \longrightarrow \mathbb{S}^1$, we have $W_{\mathcal{C}}(fg) = W_{\mathcal{C}}(f) + W_{\mathcal{C}}(g)$.

iv. If one reverses the parametrization of $\mathcal{C}$, then $W_{\leftarrow\mathcal{C}} = -W_{\mathcal{C}}$.

Definition 2.5. *Let $f: \mathcal{C} \longrightarrow \mathbb{S}^1$ be definable and continuous, and let $w_0 \in K \setminus f(\mathcal{C})$. We want to define the* **winding number** *$W_{\mathcal{C}}(f, w_0)$ of f* **around** *w_0. We define $f_{w_0}: \mathcal{C} \longrightarrow \mathbb{S}^1; z \mapsto \frac{f(z) - w_0}{|f(z) - w_0|}$, and set $W_{\mathcal{C}}(f, w_0) := W_{\mathcal{C}}(f_{w_0})$.*

2.2 Winding number and K-differentiability

Lemma 2.6. *Let $U \subseteq K$ be definable, open and non-empty. Let $f: U \longrightarrow K$ be definable and continuous, and let $z_0 \in U$ such that f is K-differentiable at z_0. If $f'(z_0) \neq 0$, then for all sufficiently small circles $\mathcal{C}$ centered at z_0, we have $W_{\mathcal{C}}(f, f(z_0)) = 1$.*

Proof. Write $d := f'(z_0) \in K^{\times}$. Since $d \neq 0$, there is a $U_1 \subseteq U$ with $f(z) \neq f(z_0)$ for all $z \in U \setminus \{z_0\}$. So we can consider $h: U_1 \longrightarrow \mathbb{S}^1; z \mapsto \frac{f(z) - f(z_0)}{|f(z) - f(z_0)|}$. Also set

$$k(z) := \frac{d\,(z - z_0)}{|d|\,|z - z_0|}.$$

Note that $W_{\mathcal{C}}(k) = 1$ for all circles $\mathcal{C}$ around z_0. Since $f'(z_0) = d$, for z sufficiently close to z_0, the element $\frac{h(z)}{k(z)}$ is close to 1. In particular, picking a sufficiently small circle $\mathcal{C}$ around z_0, the function $\frac{h}{k}: \mathcal{C} \longrightarrow \mathbb{S}^1$ is not surjective. So $W_{\mathcal{C}}(h) - W_{\mathcal{C}}(k) = W_{\mathcal{C}}\left(\frac{h}{k}\right) = 0$. We conclude that $W_{\mathcal{C}}(f, f(z_0)) = W_{\mathcal{C}}(h) = W_{\mathcal{C}}(k) = 1$. $\square$

Write D for the closed unit disk in K, and $\mathcal{C}$ for the unit circle, parametrized counter-clockwise.

Main Lemma. *Let $f: D \longrightarrow K$ be definable and K-differentiable on $\mathrm{Int}(D)$. Let $w_0 \in K \setminus \mathrm{Int}(D)$. We have*

i. If $w_0 \notin f(D)$, then $W_{\mathcal{C}}(f, w_0) = 0$.

ii. If $w_0 \in f(D)$, then $W_{\mathcal{C}}(f, w_0) > 0$.

iii. Each connected component of $K \setminus f(\mathcal{C})$ is either contained in or disjoint from $f(D)$.

Example 2.7. We give an example where ii.+iii. fail for a continuous but not K-differentiable f. Let $f(x, y) := (x, y^2)$. So $f(\mathcal{C})$ is an upper arc in K, and $K \setminus f(\mathcal{C})$ has one definably connected component, and has winding number 0 around any element in the $K \setminus f(\mathcal{C})$ including those which lie in $f(D)$.

Proof of the Main Lemma. We first prove i. We can use an homotopy to shrink $\mathcal{C}$ continuously, and as the radius of $\mathcal{C}$ tends to 0, the curve $f(\mathcal{C})$ is close to $f(0)$, so f_{w_0} will not be surjective (it will only cover a small angle). So the winding number of f_{w_0} is 0, hence the result.

Let us now prove ii. Fix a definably connected and open component X of $K \setminus f(\mathcal{C})$ which contains w_0. Since $f(D)$ is definably connected (because D is and f is continuous and definable), the point w_0 is not isolated in X, so $f(D) \cap X$ is infinite. Consider the open set $U\colon f^{-1}(X)$. We claim that the set

$$\mathcal{P} := \{z \in U : f'(z) \neq 0\}$$

has dimention 2. Indeed assume for contradiction that $\mathcal{P}$ has dimension $\leqslant 1$. Then in particular $Z := \{z \in U : f'(z) = 0\}$ has codimension $\leqslant 1$. But $f \upharpoonright Z$ has differential 0 everywhere, so partitioning Z, we see that f is locally constant, so takes only finitely many values. So $f(U)$ is finite: contradicting the previous argument.

The statements of the main lemma are first-order, so we can move to a sufficiently saturated elementary extension, and consider a generic point z_0 in X over $\varnothing$. Then f is $\mathcal{C}^1$ at z_0 as an $\mathcal{R}$-function. Moreover $J(f)_{z_0}$ is invertible (i.e. $f'(z_0) \neq 0$), then the inverse function theorem for o-minimal structures gives that $f(U) \subseteq X$ contains an open set, whence in particular $f(D) \cap X$ contains an open set.

Pick $w_1 \in f(D) \cap X$ be generic, so $\dim(w_1 / \varnothing) = 2$. We claim that $f^{-1}(\{w_1\})$ is finite and that f' is non-zero on this set. Assume for contradiction that $f^{-1}(\{w_1\})$ is infinite. Let $X_1 := \{w \in f(D) \cap X : f^{-1}(\{w\}) \text{ is infinite}\}$. Then $\dim X_1 = 2$ since it contains the generic point w_1. So $f\colon f^{-1}(\{w_1\}) \longrightarrow X_1$ is surjectve for all such infinite fibers, which is impossible since the dimentin of $f^{-1}(\{w_1\})$ is $\leqslant 0$. Assume for contradiction that there is $z \in f^{-1}(\{w_1\})$ with $f'(z) = 0$, and write X_2 for the set of such z's. Then again $X_2 := \{w \in f(D) \cap X : f'(z) = 0\}$ has dimension 2. By definable choice, there is a $Y_1 \subseteq K$ such that for all $w \in X_2$, there is a $y \in Y_1$ with $f'(y) = 0$ and this yields a similar contradiction. □

03-02: Lecture 5

Removal of singularities *à la* Riemann. *Let U be open and non-empty, let $z_0 \in U$ and let $f\colon U \setminus \{z_0\} \longrightarrow K$ be definable, K-differentiable and bounded. Then there is a unique $w_0 \in K$ such that the extension of f to U with $f(z_0) = w_0$ is K-differentiable.*

Proof. Set $h(z) := (z - z_0)\, f(z)$ for $z \in U \setminus \{z_0\}$ and $h(z_0) = 0$. Since f is bounded, we have $\lim_0 h = 0$, so h is K-differentiable on $U \setminus \{z_0\}$, so by the previous theorem, the function h is K-differentiable at z_0, with $h'(z_0) = \lim_{z_0} f$. We then extend f by continuity and see that f is K-differentiable at z_0. □

Using the previous result and the maximum principle, one can prove the following:

Theorem 2.8. *If $f\colon U \longrightarrow K$ is definable and K-differentiable, then f' is also K-differentiable.*

3 Isolated singularities

We start with an o-minimal fact:

Fact Let U be open and non-empty. Let $g\colon U \longrightarrow K$ be definable, let $z \in \overline{U}$. Then there is a neighborhood V of z such that $g(V \cap U)$ is not dense in K.

Proof. Assume for contradiction that this is not the case. So for all $w \in K \setminus f(\{z_0 \cap U\})$ there is a definable path $\gamma\colon [0;1] \longrightarrow K \setminus \{z_0\}$ converging to z_0 such that $f \circ \gamma$ tends to w. So $(\gamma, f \circ \gamma)$ tends to (z_0, w). So $\{z_0\} \times (K \setminus \{z_0\}) \subseteq \overline{f \upharpoonright U \setminus \{z_0\}} \setminus f$. But this is the frontier of a set of dimension 2, whereas $\{z_0\} \times (K \setminus \{z_0\})$ has dimension 2: a contradiction. □

We fix an open set U, a point $z_0 \in U$ and a definable and K-differentiable $f\colon U \setminus \{z_0\} \longrightarrow K$. Assume that f is not constant around z_0. We define the order $\mathrm{Ord}_{z_0}(f) \in \mathbb{Z}$ of f at z_0 as follows. Recall that for sufficiently large $r \in R^{>}$, the number $W_{\mathcal{C}_r}(f, 0)$ is constant (where $\mathcal{C}_r$ is the circle around z_0 of radius r). We then define $\mathrm{Ord}_{z_0}(f)$ to be that integer.

Case 1: z_0 is a removable singularity and $f(z_0) = 0$. Write f for the continuation on U. We have $0 \in \mathcal{D}_r := \mathrm{Conv}(\mathcal{C}_r)$, whence

$$\mathrm{Ord}_{z_0}(f) > 0,$$

by a previous theorem.

Case 2: z_0 is a removable singularity and $f(z_0) \neq 0$. Again write f for the continuation on U. If $f(z_0) \neq 0$, then we claim that

$$\mathrm{Ord}_{z_0}(f) = 0.$$

Indeed shrinking $\mathcal{C}_r$ (hence $\mathcal{D}_r$) sufficiently, we can obtain that $f(z_0)$ lie outside of $\mathcal{D}_r$.

Case 3: z_0 is not removable. In particular f must be unbounded near z_0. We claim that

$$\lim_{z_0} |f| = +\infty. \tag{3.1}$$

Let V be a neighborhood of z_0, let $w_0 \in K$ and $r_0 \in R^>$ such that

$$|f(z) - w_0| > r_0 \tag{3.2}$$

for all $z \in V \setminus \{z_0\}$. Set

$$h(z) := \frac{1}{f(z) - w_0}$$

for all $z \in V \setminus \{z_0\}$. The function h is K-differentiable, and bounded by (3.2). So z_0 is a removable singularity for h. So $\lim_{z_0} h \in K$. We conclude since f is unbounded near z_0 that $\lim_{z_0} h = 0$, whenc $|\lim_{z_0} f| = +\infty$.

Let us show that

$$\mathrm{Ord}_{z_0}(f) < 0.$$

Set

$$\eta(z) := \frac{1}{f(z)}$$

on a [épointé] neighborhood X of z_0. By (3.1), we can extend η to z_0 by setting $\eta(z_0) := 0$. Now we know that $W_{\mathcal{C}_r}(\eta, 0) > 0$, whence $W_{\mathcal{C}_r}(f, 0) = 0$ for all sufficiently small $r \in R^>$ (i.e. whenever $\mathcal{C}_r \subseteq X$).

Theorem 3.1. *Set $n := \mathrm{Ord}_{z_0}(f)$. There is a definable and K-differentiable $g: U \longrightarrow K$ with $g(z_0) \neq 0$ such that*

$$f(z) = g(z)\,(z - z_0)^n$$

for all $z \in U \setminus \{z_0\}$.

Proof. Let $r \in R^>$ be sufficiently small, so $n = W_{\mathcal{C}_r}(f, 0)$. Note that $W_{\mathcal{C}_r}((\mathrm{id} - z_0)^n, 0) = n$, so setting $g := \frac{f}{(\mathrm{id} - z_0)^n}$ on $U \setminus \{z_0\}$, we have $W_{\mathcal{C}_r}(g, 0) = 0$. By the previous trichotomy, the function g can be extended to z_0 with $g(z_0) \neq 0$; hence the result. □

Corollary 3.2. *Assume that f is K-differentiable at z_0 and that $\mathrm{Ord}_{z_0}(f) \geqslant 0$. Then $\mathrm{Ord}_{z_0}(f) := \min\{n \in \mathbb{Z} : f^{(n)}(z_0) \neq 0\}$ where $f^{(n)}(z_0) \neq 0$ means that there is an analytic continuation of f such that...*

Corollary 3.3. *For non-standard $\alpha \in R^{>\mathbb{N}}$, "the" function $z \mapsto z^\alpha$ cannot be defined as a K-differentiable map on a neighborhood of 0.*

4 Taylor series

Let $U \subseteq K$ be a non-empty open definable set, and let $z_0 \in U$.

From the proof of Theorem 3.1, we deduce:

Theorem 4.1. *Let $f: U \longrightarrow K$ be definable and K-differentiable. Then*

$$\forall n \in \mathbb{N}, \mathrm{Ord}_{z_0}\left(f - \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} (\mathrm{id} - z_0)^k \right) > n.$$

If z_0 is a pole, then

$$\forall n \geqslant \mathrm{Ord}_{z_0}(f), \mathrm{Ord}_{z_0}\left(f - \sum_{k=\mathrm{Ord}_{z_0}(f)}^{n} a_k\,(\mathrm{id} - z_0)^k \right) > 0$$

for a fixed sequence $(a_{\mathrm{Ord}_{z_0}(f)}, \ldots)$.

Corollary 4.2. *The map*

$$f \longmapsto \sum_{k \geqslant 0} \frac{f^{(k)}(z_0)}{k!}\,(z - z_0)^k$$

from germs at z_0 of K-differentiable and definable functions to power series in $K[[z - z_0]]$ is injective.

Theorem 4.3. *Every definable K-differentiable function $f: K \longrightarrow K$ is a polynomial.*

Theorem 4.4. *If $f: K \setminus \{z_0, \ldots, z_n\} \longrightarrow K$ is definable and K-differentiable, then f is a polynomial.*

5 Some model theory

Proposition 5.1. *Let $(f_t)_{t \in \Gamma}$ be a definable family of K-differentiable functions $K \longrightarrow K$, then there is an $N \in \mathbb{N}$ such that each f_t has degree $\leqslant N$.*

Question 2. Let $(f_t)_{t \in \Gamma}$ be a definable family of polynomial functions on $\mathcal{D}$. Is there a uniform bound on the degree of f_t?

Example 5.2. [by A. PIEKOSZ] Let $(a_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of complex numbers in $\mathcal{D}_{1/2}$ with absolute value $\leqslant 1/2$. Define

$$\begin{array}{rcl} f: \mathcal{D}_{1/2} \times \mathcal{D}_{1/2} & \longrightarrow & \mathbb{C} \\ (z, w) & \longmapsto & \displaystyle\sum_{n \geqslant 1} z^n\,(w - a_1) \cdots (w - a_n). \end{array}$$

This is definable in $\mathbb{R}_{\mathrm{an}}$, and for all $n \in \mathbb{N}$, the function $f(-, a_n)$ is a polynomial of degree n. But for $w \notin \{a_n : n \in \mathbb{N}\}$ the function $f(-, w)$ is not a polynomial, so this doesn't give a negative answer to the previous question.

Proposition 5.3. *Let $(f_t)_{t \in \Gamma}$ be a definable family of K-differentiable functions on $U_t \setminus \{z_t\}$ for open sets $U_t \ni z_t, t \in \Gamma$. Then there is an $N \in \mathbb{N}$, such that for all $t \in \Gamma$, either f_t is locally constant around z_t or $|\mathrm{Ord}_{z_t}(f_t)| \leqslant N$.*

Proposition 5.4. *Let $(f_t)_{t \in \Gamma}$ be a definable family of K-differentiable functions on $U_t \setminus \{z_t\}$ for open sets $U_t \ni z_t, t \in \Gamma$. For $t \in \Gamma$, let*

$$\sum_{k \geqslant -N} a_{k,t}(z_t)\,(z - z_t)^k$$

be the Laurent series associated to f_t, where N is as in the previous proposition. Then the function

$$t \mapsto a_{k,t}(z_t)$$

is definable.

Let us now go back to the classical setting $K = \mathbb{C}$. Let $\mathcal{C}$ be a simple closed curve and let f: $U \supseteq \mathcal{C} \longrightarrow K$ have finitely many residues $z_1, \ldots, z_n$ in $\mathrm{Int}(\mathrm{Hull}(\mathcal{C}))$. Recall that $\mathrm{res}_z(f) = a_{-1}(z)$ in the previous notations. We have $\frac{1}{2\pi i}\int_C f = \mathrm{res}_{z_1}(f) + \cdots + \mathrm{res}_{z_n}(f)$.

So if $(f_t)_{t \in \Gamma}$ is as above and $(\mathcal{C}_t)_{t \in \Gamma}$ is a definable family of simple closed curves, and $F_t \subseteq \mathrm{Conv}(\mathcal{C}_t)$ is finite and uniformly definable, then the function

$$t \longmapsto \int_{\mathcal{C}_t} f_t$$

is also definable.

Theorem 5.5. *If $f: K^n \longrightarrow K$ is definable and K-differentiable, then f is polynomial.*

Proof. For all $\bar{a} \in K^{n-1}$, the function $f(\bar{a}, -)$ is polynomial by the one variable corresponding result. By o-minimality, the degrees of corresponding polynomials when $\bar{a}$ ranges in K^{n-1} are bounded by some $d \in \mathbb{N}$. So $f(\bar{x}, y) = \sum_{k=0}^{d} a_k(\bar{x})\, y^k$. Loooking at $\frac{\partial f}{\partial y}$, we can conclude by induction. □

In fact, we have a result from PALAIS (1978) that for any uncountable field F and $f: F^n \longrightarrow F$ which is polynomial in each variable, the function f is in fact polynomial.

6 Behavior at boundary points

Assume that $f: U \longrightarrow K$ is definable and K-differentiable on a non-empty open set $U \subseteq K$, and let $z_0 \in \partial U$ such that $U \cap V$ is simply connected for some neighborhood V of z_0. Then $\lim_{z_0} f$ exists in $\mathbb{R} \cup \{\infty\}$. Indeed recall that for $f: \mathcal{D} \longrightarrow K$ definable, and non-constant and K-differentiable on $\mathrm{Int}(\mathcal{D})$, then $f^{\mathrm{inv}}(w)$ is finite on $\partial\mathcal{D}$. Indeed assume that f as infinitely many limit points around z_0. Then sufficiently close to z_0, one can also arrange that f is injective and that it have non-zero derivative. So the inverse map of the restriction will be K-differentiable on an open set. Then f^{inv} sends an infinite subset of $\mathcal{D}(w)$ to z_0, so f^{inv} must be constant: a contradiction.

Question 3. Assume that a definable $f: \mathcal{D} \setminus ([-1/2, 1/2] \times \{0\}) \longrightarrow \mathbb{C}$ is holomorphic and bounded. Does f extend to $\mathcal{D}$? (preserving boundedness).

7 Definable complex manifolds and analytic sets

We now work with $R = \mathbb{R}$, so $K = \mathbb{C}$. Apparently the results should still be valid in the more general context.

Definition 7.1. *A definable $\mathbb{C}$-manifold is*

i. a definable $M \subseteq \mathbb{R}^d$,

ii. a finite cover by definable subsets $U_i, i \in I$,

iii. For all $i \in I$, a definable bijection $\phi_i: U_i \longrightarrow V_i$ into an open subset V_i of $\mathbb{C}$ such that the transition maps are holomorphic.

Note that the transition maps are definable.

Example 7.2. Open subsets of $\mathbb{C}^n$, graphs of definable holomorphic maps $\mathbb{C}^n \supseteq U \longrightarrow \mathbb{C}$, as well as the projective spaces are definable manifolds. If Λ is a discrete sugroup of $(\mathbb{C}, +)$, then the quotient $\mathbb{C}/\Lambda$ can be equipped with a $\mathbb{C}$-manifold chart by realizing this within $\mathbb{C}$[7.1]. For instance, take $\Lambda = 2\pi i \mathbb{Z}$, and define $\mathbb{C}/\Lambda$ to be $\{z \in \mathbb{C}: 0 \leqslant \mathrm{Im}(z) < 2\pi\}$. We then give the two usual charts. If we are working in $\mathbb{R}_{\mathrm{an,exp}}$ in the language, then the complex exponential is definable on small strips on $\mathbb{C}$, so we can realize $\mathbb{C}/\Lambda$ as a definable "holomorphic" copy of $\mathbb{C}^{\times}$.

Fact: Every compact analytic manifold is definably biholomorphic (in the sense of manifolds) to a definable $\mathbb{C}$-manifold.

Definition 7.3. *Let M, N definable $\mathbb{C}$-manifolds. A definable holomorphic function $M \longrightarrow N$ is a definable function $f: M \longrightarrow N$ which is holomorphic through charts.*

Definition 7.4. *A definable submanifold is a definable subset $X \subseteq M$ which is an $\mathbb{R}$-submanifold, for which moreover the tangent space at each a is $\mathbb{C}$-linear.*

The only compact definale submanifolds of $\mathbb{C}$ are the finite sets.

7.1. this speaks to Lou's remark being relevant: can we not define this abstractly rather than always having to find a embedded representation?

Theorem 7.5. *Let M be a definable $\mathbb{C}$-manifold. If $X \subseteq M$ is a definable submanifold, then it has a natural structure of definable $\mathbb{C}$-manifold, and the inclusion $X \hookrightarrow M$ is a definable holomorphic function.*

Definition 7.6. *Let M be a definable $\mathbb{C}$-manifold. A definable analytic subset of M is a definable closed $X \subseteq M$ such that for all $z \in X$, there are a definale neighorhood U_z of z and finitely many definable and holomorphic functions $f_1, \ldots, f_n: U_z \longrightarrow \mathbb{C}$ with*

$$X \cap U_z = Z(f_1, \ldots, f_n) = \{y : f_1(y) = \cdots = f_n(y) = 0\}.$$

It can be showed that in fact X can be covered by finitely many such sets U_z and functions.

Example 7.7. Algeraic varieties in $\mathbb{C}^n$ are definable analytic subsets of $\mathbb{C}$. If M is a definable compact $\mathbb{C}$-manifold for $\mathbb{R}_{\text{an}}$, then every analytic subset of M is a definable analytic subset of M (again in $\mathbb{R}_{\text{an}}$).

(By Chow's theorem, every analytic subset of $\mathbb{P}^n(\mathbb{C})$ is an algebraic variety.)

8 Removal of singularities

The basic problem: M is a definable $\mathbb{C}$-manifold, we have a definable open $U \subseteq M$, and a definable analytic subset X of U as per Definition 7.6. When is the closure $\mathrm{Cl}_M(X)$ of X in M an analytic subset of M? So did we "add singularities" by taking the closure?

Example 8.1. Let $M = \mathbb{C}$ and take U to be the unit disk. So U is a definable analytic subset of itself. But the closed disk is not an analytic subset of $\mathbb{C}$.

8.1 Main removal of singularities results

We recall a classical result of removal of singularities:

Remmert-Stein theorem. *Let N be a $\mathbb{C}$-manifold, let $E \subseteq N$ be a $\mathbb{C}$-analytic subset, and set $V := N \setminus E$. If $Y \subseteq V$ is irreducible analytic subset, and $\dim_{\mathbb{C}}(Y) > \dim_{\mathbb{C}}(E)$, then $\mathrm{Cl}_N(Y)$ is an analytic subset of N.*

A bunch of *o*-minimal ROS results:

1. Assume that $\dim_{\mathbb{R}}(\mathrm{Fr}(X \cap V) \cap V) \leqslant \dim_{\mathbb{R}}(X \cap V) - 2$ for all non-empty definable open $V \subseteq M$. Then $\mathrm{Cl}_M(X)$ is an analytic susbet of M.
2. Let $E \subseteq M$ be a definable analytic subset with $U = M \setminus E$. Then $\mathrm{Cl}_M(X)$ is an analytic susbet of M.

A corollary of 1 is that

Corollary 8.2. *If $\{X_t : t \in \Gamma\}$ is a definable family of subsets of a definable $\mathbb{C}$-manifold M, then the set $\{t \in \Gamma : X_t$ is a definable analytic subset of $M\}$ is definable.*

Proof. Set $\mathrm{Reg}_{\mathbb{C}}(X_t) = \{x \in X_t :$ the germ of X_t at x is a the germ of a $\mathbb{C}$-submanifold of $M\}$ for each $t \in \Gamma$. Each $\mathrm{Reg}_{\mathbb{C}}(X_t)$ is a locally analytic definable subset of M. The set X_t is an analytic subset of M if and only if X_t is closed, if $\mathrm{Reg}_{\mathbb{C}}(X_t)$ is dense in X_t, and if $\dim_{\mathbb{R}}(\mathrm{Fr}(\mathrm{Reg}_{\mathbb{C}}(X_t) \cap V)) \leqslant \dim_{\mathbb{R}}(\mathrm{Reg}_{\mathbb{C}}(X_t) \cap V) - 2$ for all non-empty definable open $V \subseteq M$. $\square$

It follows that we have a:

Definable Chow theorem. *If $X \subseteq \mathbb{C}^n$ is a definable analytic subset, then X is algebraic.*

Corollary 8.3. *If $(X_t)_{t \in \Gamma}$ is a definable family of subsets of $\mathbb{C}^n$, then $\{t \in \Gamma : X_t$ is algebraic$\}$ is definable.*

Exercise 8.1. Show that this result fails for real-algebraic (definable) subsets in o-minimal structures.

9 Moduli spaces of elliptic curves

Assume that we have a lattice $\Lambda_\tau = \tau_1 \mathbb{Z} + \tau_2 \mathbb{Z}$ where $\tau = (\tau_1, \tau_2)$ is an $\mathbb{R}$-basis of $\mathbb{C}$. Write $F_\tau := \mathbb{C}/\Gamma_\tau$ with its definable structure of C-manifold. Then we saw that F_τ is isomorphic to an elliptic curve $E_\tau \subseteq \mathbb{P}^1(\mathbb{C})$.

We may assume that $\tau = (\tau, 1)$ where $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Recall that E_τ and $E_{\tau'}$ are isomorphic if and only if there is a $g \in \mathrm{SL}_2(\mathbb{Z})$ with $\tau' = g \cdot \tau$ (where $\cdot$ is the standard action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$). Moreover, there is a holomorphic and transcendental surjective map $j : \mathbb{H} \longrightarrow \mathbb{C}$ with

$$\forall \tau, \tau' \in \mathbb{H}, (j(\tau) = j(\tau') \Longleftrightarrow \mathrm{SL}_2(\mathbb{Z}) \cdot \tau = \mathrm{SL}_2(\mathbb{Z}) \cdot \tau' \Longleftrightarrow E_\tau \simeq E_{\tau'})$$

where the isomorphism is as abelian varieties, analytic manifolds. The function j is called the j invariant.

9.1 Fundamental domain for j

Set

$$F = \{z \in \mathbb{H} : (\mathrm{Re}(z) \in [-1/2, 0) \wedge |z| \geqslant 1) \vee (\mathrm{Re}(z) \in [0, 1/2) \wedge |z| > 1)\}.$$

Then each orbit of $\mathrm{SL}_2(\mathbb{Z})$ has excatly one representative in F. It follows that $j \restriction F : F \longrightarrow \mathbb{C}$ is still surjective (and injective).

Theorem 9.1. *The function $j \restriction F$ is definale in* $\mathbb{R}_{\mathrm{an,exp}}$.

Proof. Consider the function $e(z) := \exp(2\, i\, \pi\, z)$. Then $e \restriction F$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$ by previous results. Now on any bounded part B of F, the function j is definable on $B \cap F$ in $\mathbb{R}_{\mathrm{an}}$. For $z = x + i\, y \in \mathrm{Cl}(F)$, we have $e(z) = \exp(2\, \pi\, i\, x) \cdot \exp(-2\, \pi\, y)$, and we see that $e(\mathrm{Cl}(F))$ is the punctured disk D^* centered on 0. Recall that in particular $j(z+1) = j(z)$ for all $z \in \mathbb{H}$, and $e(z+1) = e(z)$ as well. So we can factor j by e and get an analytic map $\tilde{j} : D^* \longrightarrow \mathbb{C}$ with

$$j \restriction \mathrm{Cl}(F) = \tilde{j} \circ (e \restriction \mathrm{Cl}(F))$$

Fact: $\lim_{|z| \to +\infty} |j| = +\infty$, so $\lim_{z \to 0} |\tilde{j}| = +\infty$, i.e. 0 is a pole of $\tilde{j}$, and we can write $\tilde{j} = \frac{f}{g}$ where f, g are analytic, definable in $\mathbb{R}_{\mathrm{an}}$. So $j \restriction F$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$. $\square$